2020

MATHEMATICS

Semester-III Examination (DODL)

Paper : MATC-3.1

Linear Algebra, Special Functions, Integral Equations & Integral Transforms

Full Marks : 80 Time : 4 Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols have their usual meanings.

Block-I

[Linear Algebra]

(Marks : 30)

1. Answer any **three** questions: 10×3=30

   a) Define eigen value of a square matrix A over a field F. Show that similar matrices have the same characteristic polynomial.

   b) Show that for an $n \times n$ matrix A over a field F, the characteristic as well as minimal polynomials have the same roots, except for multiplicities. 1+4+5=10

[Turn over]

2. a) Let V be a finite dimensional vector space over the field F and $\{a_1, a_2, ..., a_n\}$ be a basis of V. If $\{b_1, b_2, ..., b_n\}$ be any arbitrary set of vectors in another vector space W over the same field F, show that there exists an unique linear transformation T from V into W such $T(a_i) = b_i$ for $i = 1,2,...,n$.

   b) Show that for a finite dimensional vector space V, the dual space of V has the same dimension as that of V. 5+5=10

3. a) Show that eigen vectors corresponding to distinct eigen values are linearly independent.

   b) Check for the diagonalizability of the matrix

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$$

4+6=10

4. a) Let T be a linear operator on a finite dimensional vector space V and λ be an eigen value of T having algebraic multiplicity m. Show that $1 \le \dim(E_\lambda) \le m$ where E_λ is the eigen space corresponding to λ.

b) Put the following matrix into Jordan form:

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$ 4+6=10

5. a) For what values of the real numbers k (if any), is the quadratic form $f(x,y) = kx^2 - 4xy + 2y^2$

i) positive, definite

ii) negative definite

b) Define direct sum of subspaces of a vector space V. Show that if $V = W_1 \oplus W_2 \oplus ... W_k$ then

i) $V = \sum_{i=1}^{k} W_i$;

ii) for any set of vectors $v_1, v_2, ...v_k$ such that $v_i \in W_i (1 \le i \le k)$ if $v_1 + v_2 + ... + v_k = 0$, then $v_i = 0$ for all i. 3+3+1+3=10

Block - II

[Special Functions]

(Marks : 20)

Answer any **four** questions: 5×4=20

6. Show that $x = \infty$ is a regular singular point of the Legendre equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

and hence obtain a series solution. 5

7. Prove that the Chebyshev's polynomial satisfies the following orthogonal property

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m=n \neq 0, \\ \pi & \text{if } m=n=0 \end{cases}$$ 5

8. a) Show that the Bessel's function satisfies the recurrence relation

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

b) Show that $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$ 3+2=5

9. a) Show that

$$P'_{n+1}(x) - 2xP_n(x) + P_{n-1}(x) = P_n(x)$$

where $P_n(x)$ is the Legendre polynomial of degree n.

b) Show that $P'_n(1) = \frac{1}{2}n(n+1)$ 3+2=5

10. a) Prove that

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right).$$

where $H_n(x)$ is the Hermite polynomial of degree n.

b) Prove that $H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}$ 3+2=5

11. a) Show that $\{T_n(x)\}^2 - T_{n-1}(x)T_{n+1}(x) = 1 - x^2$

where $T_n(x)$ is the Chebyshev's polynomial of degree n.

b) Prove that $T_n(-x) = (-1)^n T_n(x)$ 3+2=5

Block - III

[Integral Equations and Integral Transforms]

(Marks: 30)

Part - A

Marks: 15

Answer any **three** questions: 5×3=15

12. a) What do you mean by the resolvent kernel of Fredholm integral equation of second kind?

b) Determine the resolvent kernel for the Fredholm integral equation having kernel

$$K(x,t) = e^{x+t}; a = 0, b = 1$$

where, a and b respectively denotes the upper and lower limits of the integral. 1+4=5

13. Using the method of successive approximations, solve the Volterra integral equation

$$y(x) = x - \int_0^x (x-t)\,y(t)\,dt; \quad y_0(x) = 0$$ 5

14. a) Define integral equation of convolution type.

b) Let $\phi(x)$ be the solution of the integral equation

$$\phi(x) = 1 - 2x - 4x^2 + \int_0^x \left[3 + 6(x-t) - 4(x-t)^2\right]\phi(t)\,dt$$

Then find the value of $\phi(1)$. 1+4=5

15. a) Define symmetric kernel.

b) Prove that if a kernel is symmetric then all its iterated kernels are also symmetric. 1+4=5

16. Show that the integral equation

$$\phi(x) = 1 + \frac{2}{\pi}\int_0^{\pi}(\cos^2 x)\phi(t)\,dt$$

has no solution. 5

Part-B

Marks : 15

Answer any **three** questions: 5×3=15

17. a) Define function of exponential order. Show that $F(t)-e^{t^2}$ is not exponential order as $t \to \infty$.

b) Find $\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2 25}+\frac{s+4}{(s+4)^2+81}+\frac{1}{(s+2)^2+9}\right\}$

2+3=5

18. a) Prove that if $\mathcal{L}\{F(t)\}=f(s)$ and

$$G(t)=\begin{cases} F(t-a) & \text{if } t>a, \\ 0 & \text{if } t<a, \end{cases} \text{ then}$$

$$\mathcal{L}\{G(t)\}=e^{-as}f(s)$$

b) Find the Laplace transform of F(t), where

$$F(t)=\begin{cases} \cos\left(t-\frac{2\pi}{3}\right) & \text{if } t>\frac{2\pi}{3} \\ 0 & \text{if } t<\frac{2\pi}{3} \end{cases}$$

3+2=5

19. Using Laplace transformation, solve the IBVP

$$\frac{\partial\theta}{\partial\tau}=\kappa\frac{\partial^2\theta}{\partial\xi^2},\quad 0<\xi<a; t>0$$

$$I.C.:\ \theta(\xi,0)=\sin\left(\frac{\pi\xi}{a}\right),$$

$$B.C.:\ \theta(0,\tau)=0,\ \theta(a,\tau)=0$$

1+4=5

20. Find the Fourier transform of

$$f(x)=\begin{cases} 1 \text{ if } |x|<a, \\ 0 \text{ if } |x|>a. \end{cases}$$

and hence evaluate the value of

$$\int_{-\infty}^{\infty}\frac{\sin sa.\cos sx}{s}ds \text{ and } \int_0^{\infty}\frac{\sin s}{s}ds. \quad 5$$

21. Using finite Fourier sine series, determine the steady temperature $\Theta(\xi,\eta)$ in a open square $0<\xi<\pi;\ 0<\eta<\pi$. Assume that $\Theta(\xi,\eta)$ is bounded function which takes a constant value Θ_0 on the edge $\eta=\pi$ while it vanishes on the other edges of the square. 5